Calculus on Space Curves

The <u>derivative</u> of a space curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ at t is

$$\vec{r}\,'(t) := \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= \lim_{h\to 0} \left\langle \frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h}, \frac{z(t+h) - z(t)}{h} \right\rangle$$

$$= \left\langle \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}, \lim_{h\to 0} \frac{y(t+h) - y(t)}{h}, \lim_{h\to 0} \frac{z(t+h) - z(t)}{h} \right\rangle$$

$$= \langle x'(t), y'(t), z'(t) \rangle$$

Hence the <u>derivative of a space curve is computed componentwise.</u>

<u>NB</u>: This can be generalized to space curves in $\mathbb{R}^n$.

Geometrically, $\vec{r}\,'(t_0)$ yields the <u>tangent vector</u> to $\vec{r}(t)$ at time t_0.

Thus the tangent line to $\vec{r}(t)$ at time t_0 is given by

$$\vec{l}(t) = \vec{r}(t_0) + t\,\vec{r}\,'(t_0)$$

<u>Ex</u>: Compute the tangent line to the helix at $t = -\frac{\pi}{3}$.

<u>Sol</u>: $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

$$\vec{r}\,'(t) = \frac{d}{dt}\Big[\langle \cos(t), \sin(t), t \rangle\Big] = \left\langle \frac{d}{dt}[\cos(t)], \frac{d}{dt}[\sin(t)], \frac{d}{dt}[t] \right\rangle$$

$$= \langle -\sin(t), \cos(t), 1 \rangle.$$

$$\vec{r}\,'(-\tfrac{\pi}{3}) = \langle -\sin(-\tfrac{\pi}{3}), \cos(-\tfrac{\pi}{3}), 1 \rangle = \langle \tfrac{\sqrt{3}}{2}, \tfrac{1}{2}, 1 \rangle$$

$$\vec{r}(-\tfrac{\pi}{3}) = \langle \cos(-\tfrac{\pi}{3}), \sin(-\tfrac{\pi}{3}), -\tfrac{\pi}{3} \rangle = \langle \tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}, -\tfrac{\pi}{3} \rangle$$

$\therefore$ the tangent line is $\vec{l}(t) = \langle \frac{1}{2} + \frac{\sqrt{3}}{2}t, -\frac{\sqrt{3}}{2} + \frac{1}{2}t, -\frac{\pi}{3} + t \rangle$. ////

Def^n: The <u>speed</u> of space curve $\vec{r}(t)$ at time t_0 is $|\vec{r}\,'(t_0)|$.

A curve has <u>unit-speed</u> when $|\vec{r}\,'(t)| = 1$ is constant. ✗

NB: Unit-speed curves are useful. We'll see soon how to reparametrize a curve so it has unit speed...

Ex: Compute the speed of the moment curve at $t=0$.

Sol: $\vec{r}(t) = \langle t, t^2, t^3 \rangle$; $\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$$|\vec{r}\,'(t)| = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$$

$$|\vec{r}\,'(0)| = \sqrt{1 + 4\cdot 0 + 9 \cdot 0} = \sqrt{1} = 1 \text{ at time zero.}$$ ☒

Ex: Compute the (unit-speed) tangent line to the curve $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 4\cos(2t) \rangle$ at the point $(\sqrt{3}, 1, 2)$.

Sol: $\vec{r}(t) = \langle \sqrt{3}, 1, 2 \rangle \iff \begin{cases} 2\cos(t) = \sqrt{3} \\ 2\sin(t) = 1 \\ 4\cos(2t) = 2 \end{cases} \leftarrow \therefore$ use $t = \frac{\pi}{6}$

$$\vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t), -8\sin(2t) \rangle$$

$$\vec{r}\,'\left(\tfrac{\pi}{6}\right) = \langle -2\sin\left(\tfrac{\pi}{6}\right), 2\cos\left(\tfrac{\pi}{6}\right), -8\sin\left(2\cdot\tfrac{\pi}{6}\right) \rangle = \langle -1, \sqrt{3}, -4\sqrt{3} \rangle$$

$$\left|\vec{r}\,'\left(\tfrac{\pi}{6}\right)\right| = \sqrt{1^2 + \sqrt{3}^2 + 4^2\sqrt{3}^2} = \sqrt{1 + 3 + 3\cdot 16} = \sqrt{52}$$

$$\vec{v} = \frac{\vec{r}\,'\left(\frac{\pi}{6}\right)}{\left|\vec{r}\,'\left(\frac{\pi}{6}\right)\right|} = \frac{1}{\sqrt{52}} \langle -1, \sqrt{3}, -4\sqrt{3} \rangle$$

$\therefore$ tangent line is $\vec{\ell}(t) = \langle \sqrt{3}, 1, 2 \rangle + \frac{t}{\sqrt{52}} \langle -1, \sqrt{3}, -4\sqrt{3} \rangle$. ☒

Prop (Properties of Derivatives): Let $\alpha(t)$ be a scalar function and let $\vec{u}(t), \vec{v}(t)$ be space curves. Then we have

① $\frac{d}{dt}\left[\vec{v}(\alpha(t))\right] = \alpha'(t)\,\vec{v}\,'(\alpha(t))$

$= \frac{d}{dt}\left[\langle v_1(\alpha(t)), v_2(\alpha(t)), v_3(\alpha(t))\rangle\right] = \left\langle \frac{d}{dt}[v_1(\alpha(t))], \frac{d}{dt}[v_2(\alpha(t))], \frac{d}{dt}[v_3(\alpha(t))]\right\rangle$

"chain rule" $\overset{\curvearrowright}{=} \langle v_1'(\alpha(t))\,\alpha'(t), v_2'(\alpha(t))\,\alpha'(t), v_3'(\alpha(t))\,\alpha'(t)\rangle$

$= \alpha'(t)\langle v_1'(\alpha(t)), v_2'(\alpha(t)), v_3'(\alpha(t))\rangle = \alpha'(t)\,\vec{v}\,'(\alpha(t))$

② $\frac{d}{dt}\left[\vec{u}(t) + \vec{v}(t)\right] = \vec{u}\,'(t) + \vec{v}\,'(t)$

$= \frac{d}{dt}\left[\langle u_1(t)+v_1(t), u_2(t)+v_2(t), u_3(t)+v_3(t)\rangle\right]$

$= \left\langle \frac{d}{dt}[u_1(t)+v_1(t)], \frac{d}{dt}[u_2(t)+v_2(t)], \frac{d}{dt}[u_3(t)+v_3(t)]\right\rangle$

$= \langle u_1'(t)+v_1'(t), u_2'(t)+v_2'(t), u_3'(t)+v_3'(t)\rangle = \vec{u}\,'(t) + \vec{v}\,'(t).$

③ $\frac{d}{dt}\left[\alpha(t)\vec{u}(t)\right] = \alpha'(t)\vec{u}(t) + \alpha(t)\vec{u}\,'(t)$

$= \frac{d}{dt}\left[\langle \alpha(t)u_1(t), \alpha(t)u_2(t), \alpha(t)u_3(t)\rangle\right] = \left\langle \frac{d}{dt}[\alpha(t)u_1(t)], \frac{d}{dt}[\alpha(t)u_2(t)], \frac{d}{dt}[\alpha(t)u_3(t)]\right\rangle$

"product rule" $= \langle \alpha'(t)u_1(t) + \alpha(t)u_1'(t), \alpha'(t)u_2(t) + \alpha(t)u_2'(t), \alpha'(t)u_3(t) + \alpha(t)u_3'(t)\rangle$

$= \langle \alpha'(t)u_1(t), \alpha'(t)u_2(t), \alpha'(t)u_3(t)\rangle + \langle \alpha(t)u_1'(t), \alpha(t)u_2'(t), \alpha(t)u_3'(t)\rangle =$

④ $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)$

$= \frac{d}{dt}\left[u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t)\right]$

$= \frac{d}{dt}[u_1(t)v_1(t)] + \frac{d}{dt}[u_2(t)v_2(t)] + \frac{d}{dt}[u_3(t)v_3(t)]$

$= (u_1'(t)v_1(t) + u_1(t)v_1'(t)) + (u_2'(t)v_2(t) + u_2(t)v_2'(t)) + (u_3'(t)v_3(t) + u_3(t)v_3'(t))$

$= (u_1'(t)v_1(t) + u_2'(t)v_2(t) + u_3'(t)v_3(t)) + (u_1(t)v_1'(t) + u_2(t)v_2'(t) + u_3(t)v_3'(t))$

$= \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)$

dot product!

⑤ $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t)$

NB: $\frac{d}{dt}\left[\begin{vmatrix} a(t) & b(t) \\ c(t) & d(t)\end{vmatrix}\right]$

$= \frac{d}{dt}\left[\left(a(t)d(t) - b(t)c(t)\right)\right]$

$= \frac{d}{dt}\left[a(t)d(t)\right] - \frac{d}{dt}\left[b(t)c(t)\right]$

$= \left(a'(t)d(t) + a(t)d'(t)\right) - \left(b'(t)c(t) + b(t)c'(t)\right)$

$= \left(a'(t)d(t) - b'(t)c(t)\right) + \left(a(t)d'(t) - b(t)c'(t)\right)$

$= \begin{vmatrix} a'(t) & b'(t) \\ c(t) & d(t)\end{vmatrix} + \begin{vmatrix} a(t) & b(t) \\ c'(t) & d'(t)\end{vmatrix}$

now use this!

$$= \frac{d}{dt}\left[\left\langle \begin{vmatrix} u_2(t) & u_3(t) \\ v_2(t) & v_3(t)\end{vmatrix}, -\begin{vmatrix} u_1(t) & u_3(t) \\ v_1(t) & v_3(t)\end{vmatrix}, \begin{vmatrix} u_1(t) & u_2(t) \\ v_1(t) & v_2(t)\end{vmatrix}\right\rangle\right]$$

$$= \left\langle \frac{d}{dt}\begin{vmatrix} u_2(t) & u_3(t) \\ v_2(t) & v_3(t)\end{vmatrix}, -\frac{d}{dt}\begin{vmatrix} u_1(t) & u_3(t) \\ v_1(t) & v_3(t)\end{vmatrix}, \frac{d}{dt}\begin{vmatrix} u_1(t) & u_2(t) \\ v_1(t) & v_2(t)\end{vmatrix}\right\rangle$$

$$= \left\langle \begin{vmatrix} u_2'(t) & u_3'(t) \\ v_2(t) & v_3(t)\end{vmatrix} + \begin{vmatrix} u_2(t) & u_3(t) \\ v_2'(t) & v_3'(t)\end{vmatrix}, -\begin{vmatrix} u_1'(t) & u_3'(t) \\ v_1(t) & v_3(t)\end{vmatrix} - \begin{vmatrix} u_1(t) & u_3(t) \\ v_1'(t) & v_3'(t)\end{vmatrix}, \begin{vmatrix} u_1'(t) & u_2'(t) \\ v_1(t) & v_2(t)\end{vmatrix} + \begin{vmatrix} u_1(t) & u_2(t) \\ v_1'(t) & v_2'(t)\end{vmatrix}\right\rangle$$

$$= \left\langle \begin{vmatrix} u_2'(t) & u_3'(t) \\ v_2(t) & v_3(t)\end{vmatrix}, -\begin{vmatrix} u_1'(t) & u_3'(t) \\ v_1(t) & v_3(t)\end{vmatrix}, \begin{vmatrix} u_1'(t) & u_2'(t) \\ v_1(t) & v_2(t)\end{vmatrix}\right\rangle$$

$$+ \left\langle \begin{vmatrix} u_2(t) & u_3(t) \\ v_2'(t) & v_3'(t)\end{vmatrix}, -\begin{vmatrix} u_1(t) & u_3(t) \\ v_1'(t) & v_3'(t)\end{vmatrix}, \begin{vmatrix} u_1(t) & u_2(t) \\ v_1'(t) & v_2'(t)\end{vmatrix}\right\rangle$$

$$= \vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t)$$

Q: Use $|\vec{r}(t)|^2 = \vec{r}(t)\cdot\vec{r}(t)$ to compute $\frac{d}{dt}|\vec{r}(t)|$.

Sol: $\frac{d}{dt}\left[|\vec{r}(t)|^2\right] = 2\,|\vec{r}(t)|\,\frac{d}{dt}|\vec{r}(t)|$

$\qquad ||$

$\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right] = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) = 2\,\vec{r}(t)\cdot\vec{r}\,'(t)$

$\therefore$ provided $|\vec{r}(t)| \neq 0$ we obtain

$$\frac{d}{dt}|\vec{r}(t)| = \frac{1}{|\vec{r}(t)|}\,\vec{r}(t)\cdot\vec{r}\,'(t)$$

$\blacksquare$

Def^n: The <u>integral</u> of space curve $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ from $t=a$ to $t=b$ is given by (where $\Delta_n t = \frac{b-a}{n}$):

$$\int_{t=a}^{b} \vec{r}(t)\,dt := \lim_{n\to\infty} \sum_{k=0}^{n} \vec{r}(a+k\Delta_n t)\,\Delta_n t$$

$$= \lim_{n\to\infty} \left\langle \sum_{k=0}^{n} x(a+k\Delta_n t)\Delta_n t,\ \sum_{k=0}^{n} y(a+k\Delta_n t)\Delta_n t,\ \sum_{k=0}^{n} z(a+k\Delta_n t)\Delta_n t \right\rangle$$

$$= \left\langle \lim_{n\to\infty} \sum_{k=0}^{n} x(a+k\Delta_n t)\Delta_n t,\ \lim_{n\to\infty} \sum_{k=0}^{n} y(a+k\Delta_n t)\Delta_n t,\ \lim_{n\to\infty} \sum_{k=0}^{n} z(a+k\Delta_n t)\Delta_n t \right\rangle$$

$$= \left\langle \int_{t=a}^{b} x(t)\,dt,\ \int_{t=a}^{b} y(t)\,dt,\ \int_{t=a}^{b} z(t)\,dt \right\rangle$$

Hence the integral is computed componentwise.

NB: As before, the usual properties of integrals extend to this case.

Ex: Compute $\vec{r}(t)$ if $\vec{r}\,'(t) = 2t\vec{\imath} + 3t^2\vec{\jmath} + \sqrt{t}\,\vec{k}$ and $\vec{r}(1) = \vec{\imath} + \vec{\jmath}$.

Sol: $\vec{r}\,'(t) = \langle 2t, 3t^2, t^{1/2}\rangle$, $\vec{r}(1) = \langle 1,1,0\rangle$.

$$\vec{r}(t) = \int \vec{r}\,'(t)\,dt$$

$$= \left\langle \int 2t\,dt, \int 3t^2\,dt, \int t^{1/2}\,dt \right\rangle$$

$$= \left\langle t^2, t^3, \tfrac{2}{3}t^{3/2} \right\rangle + \langle c_1, c_2, c_3\rangle$$

$$\langle 1,1,0\rangle = \vec{r}(1) = \left\langle 1, 1, \tfrac{2}{3}\cdot 1\right\rangle + \langle c_1, c_2, c_3\rangle$$

$$\therefore \langle c_1, c_2, c_3\rangle = \langle 1,1,0\rangle - \left\langle 1,1,\tfrac{2}{3}\right\rangle = \left\langle 0,0,-\tfrac{2}{3}\right\rangle$$

Hence $\vec{r}(t) = \left\langle t^2, t^3, \tfrac{2}{3}t^{3/2}\right\rangle + \left\langle 0,0,-\tfrac{2}{3}\right\rangle$ ///

Frames at a Point

Recall that $\vec{i}$, $\vec{j}$, and $\vec{k}$ form the standard basis of $\mathbb{R}^3$. Given a smooth curve in $\mathbb{R}^3$ (i.e. a curve with a parametrization $\vec{r}(t)$ satisfying $|\vec{r}\,'(t)| \neq 0$ for all t), we can create frames like that which are meaningful in the context of the curve by using the derivative...

Def^n: Let $\vec{r}(t)$ be a smooth parametrization of curve C.

- The unit tangent vector to C at t_0 is $\vec{T}(t_0) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t_0)|}$
- The unit normal vector to C at t_0 is $\vec{N}(t_0) = \dfrac{\vec{T}'(t_0)}{|\vec{T}'(t_0)|}$
- The binormal vector to C at t_0 is $\vec{B}(t_0) = \vec{T}(t_0) \times \vec{N}(t_0)$

Ex: Compute the unit tangent, unit normal, and binormal of $\vec{r}(t) = \langle \cos(t), t, \sin(t) \rangle$ at $(1, 0, 0)$.

Sol:

$$\vec{r}\,'(t) = \langle -\sin(t), 1, \cos(t) \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2(t) + 1^2 + \cos^2(t)} = \sqrt{2}$$

$$T(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), 1, \cos(t) \rangle$$

$$T'(t) = \frac{1}{\sqrt{2}} \langle -\cos(t), 0, -\sin(t) \rangle$$

$$|T'(t)| = \frac{1}{\sqrt{2}} \sqrt{\cos^2(t) + 0^2 + \sin^2(t)} = \frac{1}{\sqrt{2}}$$

$$N(t) = \frac{1}{\sqrt{2}} \Big/ \frac{1}{\sqrt{2}} \langle -\cos(t), 0, -\sin(t) \rangle = \langle -\cos(t), 0, -\sin(t) \rangle$$

$$B(t) = \frac{1}{\sqrt{2}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(t) & 1 & \cos(t) \\ -\cos(t) & 0 & -\sin(t) \end{vmatrix} = \frac{1}{\sqrt{2}} \left\langle -\sin(t), -(\sin^2(t) + \cos^2(t)), \cos(t) \right\rangle$$

$$= \frac{1}{\sqrt{2}} \langle -\sin(t), -1, \cos(t) \rangle$$

$$T(0) = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle, \quad N(0) = \langle -1, 0, 0 \rangle, \quad B(0) = \frac{1}{\sqrt{2}} \langle 0, -1, 1 \rangle$$

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